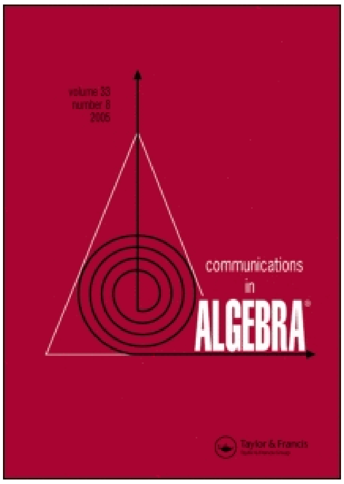

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COSET DECOMPOSITION FOR SEMISIMPLE HOPF ALGEBRAS

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The notion of double coset for semisimple finite dimensional Hopf algebras is introduced. This is done by considering an equivalence relation on the set of irreducible characters of the dual Hopf algebra. As an application formulae for the restriction of the irreducible characters to normal Hopf subalgebras are given.

Key Words: Coset decomposition; Hopf algebras; Semisimple.

2000 Mathematics Subject Classification: 16W35; 16W40.

INTRODUCTION

In this article we introduce a notion of double coset for semisimple finite dimensional Hopf algebras, similar to the one for groups. This is achieved by considering an equivalence relation on the set of irreducible characters of the dual Hopf algebra. The equivalence relation that we define generalizes the equivalence relation introduced in [7]. Using the Frobenius–Perron theory for nonnegative Hopf algebras the results from [7] are generalized and proved in a simpler manner.

The article is organized as follows. In the first section we recall some basic results for finite dimensional semisimple Hopf algebras that we need in the other sections.

In Section 2 the equivalence relation on the set of irreducible characters of the dual Hopf algebra is introduced and the coset decomposition it is proven. Using this coset decomposition in the next section we prove a result concerning the restriction of a module to a normal Hopf subalgebra. A formula for the induction from a normal Hopf subalgebra is also described using Frobenius reciprocity. A formula equivalent to the Mackey decomposition formula for groups is given in the situation of a unique double coset.

Section 4 considers 1 of the above equivalence relations but for the dual Hopf algebra. In the situation of normal Hopf subalgebras this relation can be written in terms of the restriction of the characters to normal Hopf subalgebras. Some results similar to those in group theory are proved.

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The next sections study the restriction functor from a semisimple Hopf algebra to a normal Hopf subalgebra. We define a notion of conjugate module similar to the 1 for modules over normal subgroups of a group. Some results from group theory hold in this more general setting. In particular we show that the induced module restricted back to the original normal Hopf subalgebra has as irreducible constituents the constituents of all the conjugate modules.

Algebras and coalgebras are defined over the algebraically closed ground field $k = \mathbb{C}$. For a vector space V over k by $|V|$ is denoted the dimension $\dim_k V$. The comultiplication, counit and antipode of a Hopf algebra are denoted by Δ, ϵ and S, respectively. We use Sweedler's notation $\Delta(x) = \sum x_1 \otimes x_2$ for all $x \in H$. All the other notations are those used in [6].

1. PRELIMINARIES

Throughout this article H denotes a finite dimensional semisimple Hopf algebra over $k = \mathbb{C}$. It follows that H is also cosemisimple [5]. If K is a Hopf subalgebra of H then K is also semisimple and cosemisimple Hopf algebra [6]. A Hopf subalgebra K of H is called normal Hopf subalgebra if K is stable under the left and right adjoint action, this is $h_1KS(h_2) \subset K$ and $S(h_1)Kh_2 \subset K$ for all $h \in H$.

Let $\mathcal{G}(H)$ be the Grothendieck group of the category of left H-modules. Then $\mathcal{G}(H)$ becomes a ring under the tensor product of modules and $C(H) = \mathcal{G}(H) \otimes_{\mathbb{Z}} k$ is a semisimple subalgebra of H^* [10]. Denote by $\mathrm{Irr}(H)$ the set of irreducible characters of H. Then $C(H)$ has a basis consisting of the irreducible characters $\chi \in \mathrm{Irr}(H)$. Also $C(H) = \mathrm{Cocom}(H^*)$, the space of cocommutative elements of H^*. By duality, the character ring of H^* is a semisimple subalgebra of H and under this identification it follows that $C(H^*) = \mathrm{Cocom}(H)$.

If M is an H-module with character χ then M^* is also an H-module with character $\chi^* = \chi \circ S$.

This induces an involution $^* : C(H) \to C(H)$ on $C(H)$.

For a finite dimensional semisimple Hopf algebra H use the notation $\Lambda_H \in H$ for the integral of H with $\epsilon(\Lambda_H) = |H|$ and $t_H \in H^*$ for the integral of H^* with $t_H(1) = |H|$. It follows from [6] that the regular character of H is given by the formula

$$t_H = \sum_{\chi \in \mathrm{Irr}(H)}^{s} \chi(1)\chi. \tag{1.1}$$

The dual formula is

$$\Lambda_H = \sum_{d \in \mathrm{Irr}(H^*)} \epsilon(d)d. \tag{1.2}$$

If $W \in H^*$-mod then W becomes a right H-comodule via $\rho : W \to W \otimes H$ given by $\rho(w) = \sum w_0 \otimes w_1$ if and only if $fw = \sum f(w_1)w_0$ for all $w \in W$ and $f \in H^*$.

Let W be a left H^*-module. Then W is a right H-comodule and one can associate to it a subcoalgebra of H denoted by C_W [4]. This is the minimal subcoalgebra of H with the property that $\rho(W) \subset W \otimes C_W$. If W is simple and $q = |W|$ then $|C_W| = q^2$ and the associated coalgebra C_W is a co-matrix coalgebra. It has a basis $\{x_{ij}\}_{1\le i,j\le q}$ such that $\Delta(x_{ij}) = \sum_{l=0}^{q} x_{il} \otimes x_{lj}$ for all $1 \le i, j \le q$. Moreover $W \cong k\langle x_{1i} \mid 1 \le i \le q\rangle$

as right H-comodules where $\rho(x_{1i}) = \Delta(x_{1i}) = \sum_{l=0}^{q} x_{1l} \otimes x_{li}$ for all $1 \leq i \leq q$. The character of W as left H^*-module is $d \in C(H^*) \subset H$ and it is given by $d = \sum_{i=1}^{q} x_{ii}$. Then $\epsilon(d) = q$ and the simple subcoalgebra C_W is sometimes denoted by C_d.

If M and N are right H-comodules then the tensor product $M \otimes N$ is also a right H-comodule. The associated coalgebra of $M \otimes N$ is CD where C and D are the associated subcoalgebras of M and N respectively (see [8]).

2. DOUBLE COSET FORMULA FOR COSEMISIMPLE HOPF ALGEBRAS

In this section let H be a semisimple finite dimensional Hopf algebra as before and K and L be 2 Hopf subalgebras. Then H can be decomposed as sum of $K - L$ bimodules which are free both as K-modules and L-modules and are analogues of double cosets in group theory. To the end we give an application in the situation of a unique double coset.

There is a bilinear form $m : C(H^*) \otimes C(H^*) \to k$ defined as follows (see [7]). If M and N are two H-comodules with characters c and d then $m(c, d)$ is defined as $\dim_k \mathrm{Hom}^H(M, N)$. The following properties of m (see Theorem 10 of [7]) will be used later:

$$m(x, yz) = m(y^*, zx^*) = m(z^*, x^*y) \qquad \text{and}$$
$$m(x, y) = m(y,\ x) = m(y^*, x^*)$$

for all $x, y, z \in C(H^*)$.

Let H be a finite dimensional cosemisimple Hopf algebra and K, L be two Hopf subalgebras of H. We define an equivalence relation $r_{K,L}^H$ on the set of simple subcoalgebras of H as following: $C \sim D$ if $C \subset KDL$.

Since the set of simple subcoalgebras is in bijection with $\mathrm{Irr}(H^*)$ the above relation in terms of H^*-characters becomes the following: $c \sim d$ if $m(c, \Lambda_K d \Lambda_L) > 0$ where Λ_K and Λ_L are the integrals of K and L with $\epsilon(\Lambda_K) = |K|$ and $\epsilon(\Lambda_L) = |L|$ and $c, d \in \mathrm{Irr}(H^*)$.

It is easy to see that $\sim$ is an equivalence relation. Clearly $c \sim c$ for any $c \in \mathrm{Irr}(H^*)$ since both Λ_K and Λ_L contain the trivial character.

Using the above properties of the bilinear form m, one can see that if $c \sim d$ then $m(d, \Lambda_K c \Lambda_L) = m(\Lambda_K^*, c\Lambda_L d^*) = m(c^*, \Lambda_L d^* \Lambda_K) = m(c, \Lambda_K^* d \Lambda_L^*) = m(c, \Lambda_K d \Lambda_L)$ since $\Lambda_K^* = \Lambda_K$ and $\Lambda_L^* = \Lambda_L$. Thus $d \sim c$.

The transitivity can be easier seen that holds in terms of simple subcoalgebras. Suppose that $c \sim d$ and $d \sim e$ and c, d, and e are 3 irreducible characters associated to the simple subcoalgebras C, D and E respectively. Then $C \subset KDL$ and $D \subset KEL$. The last relation implies that $KDL \subset K^2EL^2 = KEL$. Thus $C \subset KEL$ and $c \sim e$.

If $\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_l$ are the equivalence classes of $r_{K,L}^H$ on $\mathrm{Irr}(H^*)$ then let

$$a_i = \sum_{d \in \mathscr{C}_i} \epsilon(d) d \tag{2.1}$$

for $1 \leq i \leq l$.

For any character $d \in C(H^*)$ let L_d and R_d be the left and right multiplication with d on $C(H^*)$.

Proposition 2.2. *With the above notations it follows that a_i are eigenvectors of the operator $T = L_{\Lambda_K} \circ R_{\Lambda_L}$ on $C(H^*)$ corresponding to the eigenvalue $|K||L|$.*

Proof. Definition of $r^H_{K,L}$ implies that $d \sim d'$ if and only if $m(d', T(d)) > 0$. It follows that $T(a_i)$ has all the irreducible constituents in $\mathscr{C}_i$ for all $1 \leq i \leq l$. Since $\Lambda_H = \sum_{i=1}^{l} a_i$ the formula $T(\Lambda_H) = |H||K|\Lambda_H$ gives that $T(a_i) = |K||L|a_i$ for all $1 \leq i \leq l$. □

In the sequel, we use the Frobenius–Perron theorem for matrices with nonnegative entries (see [2]). If A is such a matrix then A has a positive eigenvalue λ which has the biggest absolute value among all the other eigenvalues of A. The eigenspace corresponding to λ has a unique vector with all entries positive. λ is called the principal value of A and the corresponding positive vector is called the principal vector of A. Also the eigenspace of A corresponding to λ is called the principal eigenspace of A.

The following result is also needed:

Proposition 2.3 ([2, Proposition 5]). *Let A be a matrix with nonnegative entries such that A and A^t have the same principal eigenvalue and the same principal vector. Then after a permutation of the rows and the same permutation of the columns A can be decomposed in diagonal blocks $A = A_1, A_2, \ldots, A_l$ with each block an indecomposable matrix.*

Recall from [2] that a matrix $A \in M_n(\mathbb{C})$ is called decomposable if the set $I = \{1, 2, \ldots, n\}$ can be written as a disjoint union $I = J_1 \cup J_2$ such that $a_{uv} = 0$ whenever $u \in J_1$ and $v \in J_2$. Otherwise the matrix A is called indecomposable.

Theorem 2.4. *Let H be a finite dimensional semisimple Hopf algebra and K, L be two Hopf subalgebras of H. Consider the linear operator $T = L_{\Lambda_K} \circ R_{\Lambda_L}$ on the character ring $C(H^*)$ and $[T]$ the matrix associated to T with respect to the standard basis of $C(H^*)$ given by the irreducible characters of H^*.*

(1) *The principal eigenvalue of* $[T]$ *is* $|K||L|$.
(2) *The eigenspace corresponding to the eigenvalue $|K||L|$ has $(a_i)_{1 \leq i \leq l}$ as k-basis were a_i are defined in* 2.1.

Proof. (1) Let λ be the biggest eigenvalue of T and v the principal eigenvector corresponding to λ. Then $\Lambda_K v \Lambda_L = \lambda v$. Applying ϵ on both sides of this relation it follows that $|K||L|\epsilon(v) = \lambda\epsilon(v)$. But $\epsilon(v) > 0$ since v has positive entries. It follows that $\lambda = |K||L|$.

(2) It is easy to see that the transpose of the matrix $[T]$ is also $[T]$. To check that let $x_1, \ldots, x_s$ be the basis of $C(H^*)$ given by the irreducible characters of H^* and suppose that $T(x_i) = \sum_{j=1}^{s} t_{ij}x_j$. Thus $t_{ij} = m(x_j, \Lambda_K x_i \Lambda_L)$ and $t_{ji} = m(x_i, \Lambda_K x_j \Lambda_L) = m(\Lambda_K^*, x_j \Lambda_L x_i^*) = m(x_j^*, \Lambda_L x_i^* \Lambda_K) = m(x_j, \Lambda_K^* x_i \Lambda_L^*) = t_{ij}$ since $\Lambda_K^* = S(\Lambda_K) = \Lambda_K$ and also $\Lambda_L^* = \Lambda_L$.

Proposition 2.3 implies that after a permutation of the rows and the same permutation of the columns the matrix $[T]$ decomposes in diagonal blocks

$A = \{A_1, A_2, \ldots, A_s\}$ with each block an indecomposable matrix. This decomposition of $[T]$ in diagonal blocks gives a partition $\mathrm{Irr}(H^*) = \bigcup_{i=1}^{s} \mathscr{A}_i$ on the set of irreducible characters of H^*, where each $\mathscr{A}_i$ corresponds to the rows (or columns) indexing the block A_i. The eigenspace of $[T]$ corresponding to the eigenvalue λ is the sum of the eigenspaces of the diagonal blocks $A_1, A_2, \ldots A_l$ corresponding to the same value. Since each A_i is an indecomposable matrix it follows that the eigenspace of A_i corresponding to λ is one dimensional (see [2]). If $b_j = \sum_{d\in\mathscr{A}_j} \epsilon(d)d$ then as in the proof of Proposition 2.2 it can be seen that b_j is eigenvector of T corresponding to the eigenvalue $\lambda = |K||L|$. Thus b_j is the unique eigenvector of A_j corresponding to the eigenvalue $|K||L|$. Therefore each a_i is a linear combination of these vectors. But if $d \in \mathscr{A}_i$ and $d' \in \mathscr{A}_j$ with $i \neq j$ then $m(d', T(d)) = 0$ and the definition of $r^H_{K,L}$ implies that $d \nsim d'$. This means that a_i is a scalar multiple of some b_j and this defines a bijective correspondence between the diagonal blocks and the equivalence classes of the relation $r^H_{K,L}$. Thus the eigenspace corresponding to the principal eigenvalue $|K||L|$ has a k-basis given by a_i with $1 \leq i \leq l$. □

Corollary 2.5. *Let H be a finite dimensional semisimple Hopf algebra and K, L be two Hopf subalgebras of H. Then H can be decomposed as*

$$H = \bigoplus_{i=1}^{l} B_i,$$

where each B_i is a (K, L)-bimodule free as both left K-module and right L-module.

Proof. Consider as above the equivalence relation $r^H_{K,L}$ relative to the Hopf subalgebras K and L. For each equivalence class $\mathscr{C}_i$ let $B_i = \bigoplus_{C\in\mathscr{C}_i} C$. Then $KB_iL = B_i$ from the definition of the equivalence relation. It follows that $B_i = KB_iL \in {}_K\mathscr{M}^H_L$ which implies that B_i is free as left K-module and right L-module [9]. □

The bimodules B_i from the above corollary will be called a double coset for H with respect to K and L.

Corollary 2.6. *With the above notations, if $d \in \mathscr{C}_i$ then*

$$\frac{\Lambda_K}{|K|} d \frac{\Lambda_L}{|L|} = \epsilon(d)\frac{a_i}{\epsilon(a_i)}. \tag{2.7}$$

Proof. One has that $\Lambda_K d \Lambda_L$ is an eigenvector of $T = L_{\Lambda_K} \circ R_{\Lambda_L}$ with the maximal eigenvalue $|K||L|$. From Theorem 2.4 it follows that $\Lambda_K d\Lambda_L$ is a linear combination of the elements a_j. But $\Lambda_K d\Lambda_L$ cannot contain any a_j with $j \neq i$ because all the irreducible characters entering in the decomposition of the product are in $\mathscr{C}_i$. Thus $\Lambda_K d\Lambda_L$ is a scalar multiple of a_i and the Formula 2.7 follows. □

Remark 2.8. Let C_1 and C_2 be 2 subcoalgebras of H and $K = \sum_{n\geq 0} C_1^n$ and $L = \sum_{n\geq 0} C_2^n$ be the 2 Hopf subalgebras of H generated by them [8]. The above equivalence relation $r^H_{K,L}$ can be written in terms of characters as follows: $c \sim d$ if $m(c, c_1^n d c_2^m) > 0$ for some natural numbers $m, n \geq 0$.

Remark 2.9. Setting $C_1 = k$ in the above remark Theorem 2.4 gives Theorem 7 [7]. The above equivalence relation is denoted by $r^H_{k,L}$ and can be written as $c \sim d$ if and only if $m(c, dc_2^m) > 0$ for some natural number $m \geq 0$. The equivalence class corresponding to the simple coalgebra $k1$ consists of the simple subcoalgebras of all the powers C_2^m for $m \geq 0$ that is all the simple subcoalgebras of L. Without loss of generality we may assume that this equivalence class is $\mathscr{C}_1$. It follows that $a_1 = \Lambda_L$ and

$$\frac{d}{\epsilon(d)}\frac{\Lambda_L}{|L|} = \frac{a_i}{\epsilon(a_i)}$$

for any irreducible character $d \in \mathscr{C}_i$.

Let K be a Hopf subalgebra of H and $s = |H|/|K|$. Then H is free as left K-module [9]. If $\{a_i\}_{i=1,s}$ is a basis of H as left K-module then $H = Ka_1 \oplus Ka_2 \cdots \oplus Ka_s$ as left K-modules. Consider the operator L_{Λ_K} given by left multiplication with Λ_K on H. The eigenspace corresponding to the eigenvalue $|K|$ has a basis given by $\Lambda_K a_i$, thus it has dimension s. If we restrict the operator L_{Λ_K} on $C(H^*) \subset H$ then Theorem 2.4 implies that the number of equivalence classes of $r^H_{k,K}$ is equal to the dimension of the eigenspace of L_{Λ_K} corresponding to the eigenvalue $|K|$. Thus the number of the equivalence classes of $r^H_{k,K}$ is always less or equal then the index of K in H.

Example 2.10. Let $H = kQ_8\#^{\alpha}kC_2$ be the 16-dimensional Hopf algebra described in [3]. Then $G(H^*) = \langle x\rangle \times \langle y\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathrm{Irr}(H^*)$ is given by the four 1-dimensional characters $1, x, y, xy$ and three 2-dimensional characters denoted by d_1, d_2, d_3. The algebra structure of $C(H^*)$ is given by:

$$x.d_1 = d_3 = d_1.x, \qquad x.d_2 = d_2 = d_2.x, \qquad x.d_3 = d_1 = d_3.x$$
$$y.d_i = d_i = d_i.y \qquad \text{for all } i = 1, 3$$
$$d_1^2 = d_3^2 = x + xy + d_2, \qquad d_2^2 = 1 + x + y + xy, \qquad d_1d_2 = d_2d_1 = d_1 + d_3$$
$$d_1d_3 = d_3d_1 = 1 + y + d_2$$

Consider $K = k\langle x\rangle$ as Hopf subalgebra of H and the equivalence relation $r^H_{k,K}$ on $\mathrm{Irr}(H^*)$. The equivalence classes are given by $\{1, x\}$, $\{y, xy\}$, $\{d_2\}$ and $\{d_1, d_3\}$ and the number of them is strictly less than the index of K and H. If $C_2 \subset H$ is the coalgebra associated to d_2 then the third equivalence class gives in the decomposition 2.5 the free K-module $C_2K = C_2$ whose rank is strictly less then the dimension of C_2.

3. MORE ON COSET DECOMPOSITION

Let H be a semisimple Hopf algebra and A be a Hopf subalgebra. Define $H//A = H/HA^+$ and let $\pi : H \to H//A$ be the natural module projection. Since HA^+ is a coideal of H it follows that $H//A$ is a coalgebra and π is also a coalgebra map.

Let k be the trivial A-module via the counit ϵ. It can be checked that $H//A \cong H \otimes_A k$ as H-modules via the map $\hat{h} \mapsto h \otimes_A 1$. Thus $\dim_k H//A = \mathrm{rank}_A H$.

If L and K are Hopf subalgebras of H define $LK//K := LK/LK^+$. LK is a right free K-module since $LK \in \mathcal{M}_K^H$. A similar argument to the 1 above shows that $LK//K \cong LK \otimes_K k$ as left L-modules where k is the trivial K-module. Thus $\dim_k LK//K = \mathrm{rank}_K LK$. It can be checked that LK^+ is a coideal in LK and therefore $LK//K$ has a natural coalgebra structure.

Theorem 3.1. *Let H be a semisimple Hopf algebra and K, L be 2 Hopf subalgebras of H. Then $L//L \cap K \cong LK//K$ as coalgebras and left L-modules.*

Proof. Define the map $\phi : L \to LK//K$ by $\phi(l) = \hat{l}$. Then ϕ is the composition of $L \hookrightarrow LK \to LK//K$ and is a coalgebra map as well as a morphism of left L-modules. Moreover ϕ is surjective since $\hat{lk} = \epsilon(k)\hat{l}$ for all $l \in L$ and $k \in K$. Clearly $L(L \cap K)^+ \subset \ker(\phi)$ and thus ϕ induces a surjective map $\phi : L//L \cap K \to LK//K$.

Next it will be shown that

$$\frac{|L|}{|L \cap K|} = \frac{|LK|}{|K|},$$

which implies that ϕ is bijective since both spaces have the same dimension. Consider on $\mathrm{Irr}(H^*)$ the equivalence relation introduced above and corresponding to the linear operator $L_{\Lambda_L} \circ R_{\Lambda_K}$. Assume without loss of generality that $\mathscr{C}_1$ is the equivalence class of the character 1 and put $d = 1$ the trivial character, in the Formula 2.7. Thus $\frac{\Lambda_L}{|L|}\frac{\Lambda_K}{|K|} = \frac{a_1}{\epsilon(a_1)}$. But from the definition of $\sim$ it follows that a_1 is formed by the characters of the coalgebra LK. On the other hand $\Lambda_L = \sum_{d \in \mathrm{Irr}(L^*)} \epsilon(d)d$ and $\Lambda_K = \sum_{d \in \mathrm{Irr}(K^*)} \epsilon(d)d$ (see [4]). Equality 3.2 follows counting the multiplicity of the irreducible character 1 in $\Lambda_K \Lambda_L$. Using Theorem 10 of [7] we know that $m(1, dd') > 0$ if and only if $d' = d^*$ in which case $m(1, dd') = 1$. Then $m\big(1, \frac{\Lambda_L}{|L|}\frac{\Lambda_K}{|K|}\big) = \frac{1}{|L||K|} \sum_{d \in \mathrm{Irr}(L \cap K)} \epsilon(d)^2 = \frac{|L \cap K|}{|L||K|}$ and $m\big(1, \frac{a_1}{\epsilon(a_1)}\big) = \frac{1}{\epsilon(a_1)} = \frac{1}{|LK|}$. □

Corollary 3.2. *If K and L are Hopf subalgebras of H then* $\mathrm{rank}_K LK = \mathrm{rank}_{L \cap K} L$.

Proposition 3.3. *Let H be a finite dimensional cosemisimple Hopf algebra and K, L be 2 Hopf subalgebras of H such that $KL = LK$. If M is a K-module then*

$$M \uparrow_K^{LK} \downarrow_L \cong (M \downarrow_{L \cap K}) \uparrow^L$$

as left L-modules.

Proof. For any K-module M one has

$$M \uparrow^{LK} \downarrow_L = LK \otimes_K M$$

while

$$(M \downarrow_{L \cap K}) \uparrow^L = L \otimes_{L \cap K} M.$$

The previous Corollary implies that $\mathrm{rank}_K LK = \mathrm{rank}_{L \cap K} L$ thus both modules above have the same dimension.

Define the map $\phi : L \otimes_{L\cap K} M \to LK \otimes_K M$ by $\phi(l \otimes_{L\cap K} m) = l \otimes_K m$ which is the composition of $L \otimes_{L\cap K} M \hookrightarrow LK \otimes_{L\cap K} M \to LK \otimes_K M$. Clearly ϕ is a surjective homomorphism of L-modules. Equality of dimensions implies that ϕ is an isomorphism. □

If $LK = H$ then the previous theorem is the generalization of Mackey's theorem decomposition for groups in the situation of a unique double coset.

4. A DUAL RELATION

Let K be a normal Hopf subalgebra of H and $L = H//K$. Then the natural projection $\pi : H \to L$ is a surjective Hopf map and then $\pi^* : L^* \to H^*$ is an injective Hopf map. We identify L^* with its image $\pi^*(L^*)$ in H^*. This is a normal Hopf subalgebra of H^*. In this section we will study the equivalence relation $r^{H^*}_{L^*,k}$ on $\mathrm{Irr}(H^{**}) = \mathrm{Irr}(H)$.

The following result was proven in [1].

Proposition 4.1. *Let K be a normal Hopf subalgebra of a finite dimensional semisimple Hopf algebra H and $L = H//K$. If $t_L \in L^*$ is the integral on L with $t_L(1) = |L|$ then $\epsilon_K \uparrow^H_K = t_L$ and $t_L \downarrow^H_K = \frac{|H|}{|K|}\epsilon_K$.*

Proposition 4.2. *Let K be a normal Hopf subalgebra of a semisimple Hopf algebra H and $L = H//K$. Consider the equivalence relation $r^{H^*}_{L^*,k}$ on $\mathrm{Irr}(H)$. Then $\chi \sim \mu$ if and only if their restrictions to K have a common constituent.*

Proof. The equivalence relation $r^{H^*}_{L^*,k}$ on $\mathrm{Irr}(H)$ becomes the following: $\chi \sim \mu$ if and only if $m_H(\chi, t_L\mu) > 0$. On the other hand, applying the previous Proposition it follows that:

$$\begin{aligned} m_H(\chi, t_L\mu) &= m_H(t_L^*, \mu\chi^*) = m_H(t_L, \mu\chi^*) \\ &= m_H(\epsilon \uparrow^H_K, \mu\chi^*) = m_K(\epsilon_K, (\mu\chi^*) \downarrow_K) \\ &= m_K(\epsilon_K, \mu \downarrow_K \chi^* \downarrow_K) = m_K(\chi \downarrow_K, \mu \downarrow_K). \end{aligned}$$

Thus $\chi \sim \mu$ if and only if their restriction to K have a common constituent. □

Theorem 4.3. *Let K be a normal Hopf subalgebra of a semisimple Hopf algebra H and $L = H//K$. Consider the equivalence relation $r^{H^*}_{L^*,k}$ on $\mathrm{Irr}(H)$. Then $\chi \sim \mu$ if and only if $\frac{\chi\downarrow_K}{\chi(1)} = \frac{\mu\downarrow_K}{\mu(1)}$.*

Proof. Let $\mathscr{C}_1, \mathscr{C}_2, \dots \mathscr{C}_l$ be the equivalence classes of $r^{H^*}_{L^*,k}$ on $\mathrm{Irr}(H)$ and let

$$a_i = \sum_{\chi\in\mathscr{C}_i} \chi(1)\chi \tag{4.4}$$

for $1 \leq i \leq l$. If $\mathscr{C}_1$ is the equivalence class of the trivial character ϵ then the definition of $r^{H^*}_{L^*,k}$ implies that $a_1 = t_L$. Formula from Remark 2.9 becomes

$$\frac{t_L}{|L|}\frac{\chi}{\chi(1)} = \frac{a_i}{a_i(1)}$$

for any irreducible character $\chi \in \mathscr{C}_i$.

Restriction to K of the above relation combined with Proposition 4.1 gives:

$$\frac{\chi\downarrow_K}{\chi(1)} = \frac{a_i\downarrow_K}{a_i(1)}. \tag{4.5}$$

Thus if $\chi \sim \mu$ then $\frac{\chi\downarrow_K}{\chi(1)} = \frac{\mu\downarrow_K}{\mu(1)}$. □

4.1. Formulae for Restriction and Induction

The previous theorem implies that the restriction of 2 irreducible H-characters to K either have the same common constituents or they have no common constituents. Let t_H be the integral on H with $t_H(1) = |H|$. One has that $t_H = \sum_{i=1}^{l} a_i$ as t_H is the regular character of H. Since H is free as K-module [9] it follows that the restriction of t_H to K is the regular character of K multiplied by $|H|/|K|$. Thus $t_H\downarrow_K = |H|/|K|t_K$. But $t_K = \sum_{\alpha\in\mathrm{Irr}(K)}\alpha(1)\alpha$ and Theorem 4.3 implies that the set of the irreducible characters of K can be partitioned in disjoint subsets $\mathscr{A}_i$ with $1 \leq i \leq l$ such that

$$a_i\downarrow_K = \frac{|H|}{|K|}\sum_{\alpha\in\mathscr{A}_i}\alpha(1)\alpha.$$

Then if $\chi \in \mathscr{C}_i$ formula 4.5 implies that

$$\chi\downarrow_K = \frac{\chi(1)}{a_i(1)}\frac{|H|}{|K|}\sum_{\alpha\in\mathscr{A}_i}\alpha(1)\alpha.$$

Let $|\mathscr{A}_i| = \sum_{\alpha\in\mathscr{A}_i}\alpha^2(1)$. Evaluating at 1 the above equality one gets $a_i(1) = \frac{|H|}{|K|}|\mathscr{A}_i|$. By Frobenius reciprocity the above restriction formula implies that if $\alpha \in \mathscr{A}_i$ then

$$\alpha\uparrow_K^H = \frac{\alpha(1)}{a_i(1)}\frac{|H|}{|K|}\sum_{\chi\in\mathscr{C}_i}\chi(1)\chi. \tag{4.6}$$

5. RESTRICTION OF MODULES TO NORMAL HOPF SUBALGEBRAS

Let G be a finite group and H a normal subgroup of G. If M is an irreducible H-module then

$$M\uparrow_H^G\downarrow_H^G = \bigoplus_{i=1}^{s}{}^{g_i}M,$$

where ${}^{g}M$ is a conjugate module of M and $\{g_i\}_{i=1,s}$ is a set of representatives for the left cosets of H in G. For $g \in G$ the H-module ${}^{g}M$ has the same underlying vector space as M and the multiplication with $h \in H$ is given by $h.n = (g^{-1}hg)n$ for all $n \in N$. It is easy to see that $gN \cong g'N$ if $gN = g'N$.

Let K be a normal Hopf subalgebra of H and M be an irreducible K-module. In this section we will define the notion of a conjugate module to M similar to group situation. If $d \in \mathrm{Irr}(H^*)$ we define a conjugate module ${}^dM$. The left cosets of K in H correspond to the equivalence classes of $r^H_{K,k}$. We will show that if d, d' are 2 irreducible characters in the same equivalence class of $r^H_{K,k}$ then the modules ${}^dM$ and ${}^{d'}M$ have the same constituents. We will show that the irreducible constituents of $M \uparrow^H_K \downarrow^H_K$ and $\bigoplus^d_{d \in \mathrm{Irr}(H^*)} M$ are the same.

Remark 5.1. Since K is a normal Hopf subalgebra it follows that Λ_K is a central element of H (see [1]) and by their definition $r^H_{K,k} = r^H_{k,K}$. Thus the left cosets are the same with the right cosets in this situation.

Let K be a normal Hopf subalgebra of H and M be a K-module. If W is an H^*-module then $W \otimes M$ becomes a K-module with

$$k(w \otimes m) = w_0 \otimes (S(w_1)kw_2)m. \tag{5.2}$$

In order to check that $W \otimes M$ is a K-module one has that

$$\begin{aligned} k.(k'(w \otimes m)) &= k(w_0 \otimes (S(w_1)k'w_2)m) \\ &= w_0 \otimes (S(w_1)kw_2)((S(w_3)k'w_4)m) \\ &= w_0 \otimes (S(w_1)kk'w_2)m \\ &= (kk')(w \otimes m) \end{aligned}$$

for all $k, k' \in K$, $w \in W$ and $m \in M$.

It can be checked that if $W \cong W'$ as H^*-modules then $W \otimes M \cong W' \otimes M$. Thus for any irreducible character $d \in \mathrm{Irr}(H^*)$ associated to a simple H-comodule W one can define the K-module ${}^dM \cong W \otimes M$.

Proposition 5.3. *Let K be a normal Hopf subalgebra of H and M be an irreducible K-module with character $\alpha \in C(K)$. Suppose that W is a simple H^*-module with character $d \in \mathrm{Irr}(H^*)$. Then the character α_d of the K-module ${}^dM$ is given by the following formula*:

$$\alpha_d(x) = \alpha(Sd_1xd_2)$$

for all $x \in K$.

Proof. Indeed one may suppose that $W \cong k\langle x_{1i} \mid 1 \le i \le q\rangle$ where $C_d = k\langle x_{ij} \mid 1 \le i, j \le q\rangle$ is the coalgebra associated to W and $q = \epsilon(d) = |W|$. Then formula 5.7 becomes $k(x_{1i} \otimes w) = \sum_{j,l=1}^q x_{1j} \otimes (S(x_{jl})kx_{lj})m$. Since $d = \sum_{i=1}^q x_{ii}$ one gets the the formula for the character α_d. $\square$

For any $d \in \mathrm{Irr}(H^*)$ define the linear operator $c_d : C(K) \to C(K)$ which on the basis given by the irreducible characters is given by $c_d(\alpha) = \alpha_d$ for all $\alpha \in \mathrm{Irr}(K)$.

Remark 5.4. From the above formula it can be directly checked that ${}^{dd'}\alpha = {}^{d}\,({}^{d'}\alpha)$ for all $d, d' \in \mathrm{Irr}(H^*)$ and $\alpha \in C(K)$. This shows that $C(K)$ is a left $C(H^*)$-module. Also one can verify that ${}^{d}(\alpha^*) = ({}^{d}\alpha)^*$.

Proposition 5.5. *Let K be a normal Hopf subalgebra of H and M be an irreducible K-module with character $\alpha \in C(K)$. If $d, d' \in \mathrm{Irr}(H^*)$ lie in the same coset of $r^H_{K,k}$ then ${}^{d}M$ and ${}^{d'}M$ have the same irreducible constituents. Moreover $\frac{\alpha_d}{\epsilon(d)} = \frac{\alpha_{d'}}{\epsilon(d')}$.*

Proof. Consider the equivalence relation $r^H_{k,K}$ from Section 2 and $H = \bigoplus_{i=1}^{s} B_i$ the decomposition from Corollary 2.5. Let $\mathcal{B}_1, \dots, \mathcal{B}_s$ be the equivalence classes and let b_i be defined as in 2.1. Then Formula 2.9 becomes

$$\frac{d}{\epsilon(d)}\frac{\Lambda K}{|K|} = \frac{b_i}{\epsilon(b_i)}, \tag{5.6}$$

where Λ_K is the integral in K with $\epsilon(\Lambda_K) = |K|$ and $d \in B_i$. Thus

$$\begin{aligned}
\alpha_{b_i}(x) &= \alpha(S(b_i)_1 x (b_i)_2)\\
&= \frac{\epsilon(b_i)}{\epsilon(d)|K|}\alpha(S(d\Lambda_K)_1)x(d\Lambda_K)_2)\\
&= \frac{\epsilon(b_i)}{\epsilon(d)|K|}\alpha(S((\Lambda_K)_1)S(d_1)xd_2(\Lambda_K)_2)\\
&= \frac{\epsilon(b_i)}{\epsilon(d)}\alpha(Sd_1xd_2)\\
&= \frac{\epsilon(b_i)}{\epsilon(d)}\alpha_d(x)
\end{aligned}$$

for all $d \in \mathcal{B}_i$.

This implies that if $d \sim d'$ then $\frac{\alpha_d}{\epsilon(d)} = \frac{\alpha_{d'}}{\epsilon(d')}$. □

Let N be a H-module and W an H^*-module. Then $W \otimes N$ becomes an H-module such that

$$h(w \otimes m) = w_0 \otimes (S(w_1)hw_2)m. \tag{5.7}$$

It an be checked that $W \otimes N \cong N^{|W|}$ as H-modules. Indeed the map $\phi : W \otimes N \to {}_{\epsilon}W \otimes N$ $w \otimes n \mapsto w_0 \otimes w_1 n$ is an isomorphism of H-modules where ${}_{\epsilon}W$ is considered left H-module with the trivial action. Its inverse is given by $w \otimes m \mapsto w0 \otimes S(w_1)m$. To check that ϕ is an H-module map one has that

$$\begin{aligned}
\phi(h.(w \otimes n)) &= \phi(w_0 \otimes (S(w_1)hw_2)n)\\
&= w_0 \otimes w_1(S(w_2)hw_3)n\\
&= w_0 \otimes hw_1 n\\
&= h.(w_0 \otimes w_1 n)\\
&= h\phi(w \otimes n)
\end{aligned}$$

for all $w \in W$, $m \in M$ and $h \in H$.

Proposition 5.8. *Let K be a normal Hopf subalgebra of H and M be an irreducible K-module with character $\alpha \in C(K)$. If $d \in \mathrm{Irr}(H^*)$ then*

$$\frac{1}{\epsilon(d)}\alpha_d \uparrow_K^H = \alpha \uparrow_K^H .$$

Proof. Using the notations from Subsection 4.1 let $\mathscr{A}_i$ be the subset of $\mathrm{Irr}(K)$ which contains α. It is enough to show that the constituents of α_d are contained in this set and then the induction Formula 4.6 from the same subsection can be applied. For this, suppose N is an irreducible H-module and

$$N \downarrow_K = \bigoplus_{i=1}^{s} N_i, \tag{5.9}$$

where N_i are irreducible K-modules. The above result implies that $W \otimes N \cong N^{|W|}$ as H-modules. Therefore $(W \otimes N) \downarrow_K = (N \downarrow_K)^{|W|}$ as K-modules. But $(W \otimes N) \downarrow_K = \bigoplus_{i=1}^{s}(W \otimes N_i)$ where each $W \otimes N_i$ is a K-module by 5.1. Thus

$$\bigoplus_{i=1}^{s} N_i^{|W|} = \bigoplus_{i=1}^{s}(W \otimes N_i). \tag{5.10}$$

This shows that if N_i is a constituent of $N \downarrow_K$ then $W \otimes N_i$ has all the irreducible K- constituents among those of $N \downarrow_K$. The Formula 4.6 applied for each irreducible constituent of α_d gives that

$$\frac{1}{\epsilon(d)}\alpha_d \uparrow_K^H = \alpha \uparrow_K^H \tag{5.11}$$

for all $\alpha \in \mathrm{Irr}(K)$ and $d \in \mathrm{Irr}(H^*)$. □

Proposition 5.12. *Let K be a normal Hopf subalgebra of H and M be an irreducible K-module. Then $M \uparrow_K^H \downarrow_K^H$ and $\bigoplus_{d \in \mathrm{Irr}(H^*)}^{d} M$ have the same irreducible constituents.*

Proof. Consider the equivalence relation $r_{k,K}^H$ from Section 2 and let $\mathscr{B}_1, \ldots, \mathscr{B}_s$ be its equivalence classes. Pick an irreducible character $d_i \in \mathscr{B}_i$ in each equivalence class of $r_{k,K}^H$ and let C_i be its associated simple coalgebra. Then Corollary 2.5 implies that $H = \bigoplus_{i=1}^{s} C_i K$. It follows that the induced module $M \uparrow_K^H$ is given by

$$M \uparrow_K^H = H \otimes_K M = \bigoplus_{i=1}^{s} C_i K \otimes_K M.$$

Each $C_i K \otimes_K M$ is a K-module by left multiplication with elements of K since

$$k.(ck' \otimes_K m) = c_1(Sc_2kc_3)k' \otimes_K m = c_1 \otimes_K (Sc_2kc_3)k'm$$

for all $k, k' \in K$, $c \in C_i$ and $m \in M$. Thus $M \uparrow_K^H$ restricted to K is the sum of the K-modules $C_i K \otimes_K M$. On the other hand the composition of the canonical maps $C_i \otimes K \hookrightarrow C_i K \otimes M \to C_i K \otimes_K M$ is a surjective morphism of K-modules which implies that $C_iK \otimes_K M$ is a homeomorphic image of $\epsilon(d_i)$ copies of ${}^{d_i}M$. Therefore the irreducible constituents of $M \uparrow_K^H \downarrow_K^H$ are among those of $\bigoplus_{d \in \mathrm{Irr}(H^*)}^{d} M$. In the proof of the previous Proposition we showed the other inclusion. Thus $M \uparrow_K^H \downarrow_K^H$ and $\bigoplus_{d \in \mathrm{Irr}(H^*)}^{d} M$ have the same irreducible constituents. □

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